

SPECTRUM OF A FAMILY OF OPERATORS

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Abstract. Having as start point the classic definitions of resolvent set and spectrum of a linear bounded operator on a Banach space, we introduce the resolvent set and spectrum of a family of linear bounded operators on a Banach space. In addition, we present some results which adapt to asymptotic case the classic results.

1 Introduction

Let X be a complex Banach space and $L(X)$ the Banach algebra of linear bounded operators on X . Let T be a linear bounded operator on X . The *norm* of T is

$$\|T\| = \sup \{ \|Tx\| \mid x \in X, \|x\| \leq 1 \}.$$

The *spectrum* of an operator $T \in L(X)$ is defined as the set

$$Sp(T) = \mathbb{C} \setminus r(T),$$

where $r(T)$ is the *resolvent set* of T and consists in all complex numbers $\lambda \in \mathbb{C}$ for which the operator $\lambda I - T$ is bijective on X .

It is an important fact that the *resolvent function* $\lambda \mapsto (\lambda I - T)^{-1}$ is an analytic function from $r(T)$ to $L(X)$ and for $\lambda \in r(T)$ we have

$$d(\lambda, r(T)) \geq \frac{1}{\|(\lambda I - T)^{-1}\|}.$$

Moreover, for $\lambda \in r(T)$, the *resolvent operator* $R(\lambda, T) \in L(X)$ is defined by the relation $R(\lambda, T) = (\lambda I - T)^{-1}$ and satisfied the *resolvent equation*

$$R(\lambda, T) - R(\mu, T) = (\mu - \lambda)R(\lambda, T)R(\mu, T),$$

for all $\lambda, \mu \in r(T)$. Therefore, in particular, $R(\lambda, T)$ and $R(\mu, T)$ commute.

We say that an infinite series of operators $\sum T_n$ is absolutely convergent if the series $\sum \|T_n\|$ is convergent in $L(X)$ and $\|\sum T_n\| \leq \sum \|T_n\|$.

If $\|T\| < 1$, then

$$(\lambda I - T)^{-1} = \sum T^n$$

and it is absolutely convergent. A consequence of this is the fact that $r(T)$ is an open set of \mathbb{C} .

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Theorem 1. *Theorem 1.1. Let $T \in L(X)$ be a linear bounded operator on X . Then $Sp(T)$ is a non-empty compact subset of \mathbb{C} .*

The spectral radius of an operator $T \in L(X)$ is the positive number equal with $\sup_{\lambda \in Sp(T)} |\lambda|$.

Theorem 2. *Let $T \in L(X)$. Then*

$$\sup_{\lambda \in Sp(T)} |\lambda| = \lim_{n \rightarrow \infty} \|T^n\|^{\frac{1}{n}}.$$

Let Ω be an open neighborhood of $Sp(T)$ and let $H(\Omega)$ denote the space of all complex valued analytic functions defined on Ω . The application $f \mapsto f(T) : H(\Omega) \rightarrow L(X)$ defined by the relation

$$f(T) = \frac{1}{2\pi i} \int_{\gamma} f(\lambda) R(\lambda, T) d\lambda,$$

where γ is a contour which envelopes $Sp(T)$ in Ω , is called the holomorphic functional calculi of T .

Theorem 3. *Let $T \in L(X)$ and suppose that Ω is an open neighborhood of $Sp(T)$. Then, for all $f \in H(\Omega)$, we have*

$$f(Sp(T)) = Sp(f(T)).$$

We also remember that two operators $T, S \in L(X)$ are *quasinilpotent equivalent* if

$$\lim_{n \rightarrow \infty} \left\| (T - S)^{[n]} \right\|^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \left\| (S - T)^{[n]} \right\|^{\frac{1}{n}} = 0,$$

where $(T - S)^{[n]} = \sum_{k=0}^n (-1)^{n-k} C_k^n T^k S^{n-k}$, for any $n \in \mathbb{N}$.

The quasinilpotent equivalence relation is reflexive and symmetric. It is also transitive on $L(X)$.

Theorem 4. *Theorem 1.4. Let $T, S \in L(X)$ be two quasinilpotent equivalent operators. Then*

$$Sp(T) = Sp(S).$$

2 Asymptotic equivalence and asymptotic quasinilpotent equivalence

Definition 5. We say that two families of operators $\{S_h\}, \{T_h\} \subset L(X)$, with $h \in (0, 1]$, are asymptotically equivalent if

$$\lim_{h \rightarrow 0} \|S_h - T_h\| = 0.$$

Two families of operators $\{S_h\}, \{T_h\} \subset L(X)$, with $h \in (0, 1]$, are asymptotically quasinilpotent equivalent if

$$\lim_{n \rightarrow \infty} \limsup_{h \rightarrow 0} \left\| (S_h - T_h)^{[n]} \right\|^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \limsup_{h \rightarrow 0} \left\| (T_h - S_h)^{[n]} \right\|^{\frac{1}{n}} = 0.$$

Proposition 6. The asymptotic (quasinilpotent) equivalence between two families of operators $\{S_h\}, \{T_h\} \subset L(X)$ is an equivalence relation (i.e. reflexive, symmetric and transitive) on $L(X)$.

Proof. It is evidently that the asymptotic equivalence is reflexive and symmetric. Let $\{S_h\}, \{T_h\}, \{U_h\} \subset L(X)$ be families of linear bounded operators such that $\{S_h\}, \{T_h\}$ and $\{U_h\}, \{T_h\}$ are respectively asymptotically equivalent. Then

$$\begin{aligned} & \limsup_{h \rightarrow 0} \|S_h - U_h\| \\ &= \limsup_{h \rightarrow 0} \|S_h - T_h + T_h - U_h\| \leq \lim_{h \rightarrow 0} \|S_h - T_h\| + \lim_{h \rightarrow 0} \|T_h - U_h\| \\ &= 0. \end{aligned}$$

The asymptotic quasinilpotent equivalence is also reflexive and symmetric. In order to prove that it is transitive, let $\{S_h\}, \{T_h\}, \{P_h\} \subset L(X)$ such that $\{T_h\}, \{P_h\}$ and $\{S_h\}, \{P_h\}$ be respectively asymptotically quasinilpotent equivalent. Then for any $\varepsilon > 0$ there exists a $n_\varepsilon \in \mathbb{N}$ such that

$$(T_h - P_h)^{[j]} < \varepsilon^j$$

and

$$(P_h - S_h)^{[n-j]} < \varepsilon^{n-j},$$

for every $j, n - j > n_\varepsilon$ and $h \in (0, 1]$.

Taking

$$M_\varepsilon = \max_{1 \leq j \leq n_\varepsilon} \left\{ \frac{\|(T_h - P_h)^{[j]}\|}{\varepsilon^j}, \frac{\|(P_h - S_h)^{[j]}\|}{\varepsilon^j}, 1 \right\}$$

we obtain

$$\|(T_h - P_h)^{[j]}\| < \varepsilon^j M_\varepsilon$$

and

$$\|(P_h - S_h)^{[j]}\| < \varepsilon^j M_\varepsilon,$$

for every $j \in \mathbb{N}$ and $h \in (0, 1]$.

In view of above inequality and the following equality

$$(T - S)^{[n]} = \sum_{k=0}^n (-1)^{n-k} C_n^k (T - P)^{[k]} (P - S)^{[n-k]},$$

for every $n \in \mathbb{N}$ and $P \in L(X)$, it results that

$$\begin{aligned} \|(T_h - S_h)^{[n]}\| &\leq \sum_{k=0}^n C_n^k \|(T_h - P_h)^{[k]}\| \|(P_h - S_h)^{[n-k]}\| \\ &\leq \sum_{k=0}^n C_n^k \varepsilon^k \varepsilon^{n-k} M_\varepsilon^2 \\ &= (2\varepsilon)^n M_\varepsilon^2, \end{aligned}$$

for every $n \in \mathbb{N}$ and $h \in (0, 1]$.

Therefore

$$\limsup_{h \rightarrow 0} \|(T_h - S_h)^{[n]}\| \leq (2\varepsilon)^n M_\varepsilon^2$$

and thus

$$\limsup_{h \rightarrow 0} \|(T_h - S_h)^{[n]}\|^{\frac{1}{n}} \leq 2\varepsilon M_\varepsilon^{2/n}.$$

Consequently

$$\lim_{n \rightarrow \infty} \limsup_{h \rightarrow 0} \|(T_h - S_h)^{[n]}\|^{\frac{1}{n}} \leq 2\varepsilon,$$

for any $\varepsilon > 0$.

Analogously we prove that $\lim_{n \rightarrow \infty} \limsup_{h \rightarrow 0} \|(S_h - T_h)^{[n]}\|^{\frac{1}{n}} = 0$. \square

Proposition 7. *Let $\{S_h\}, \{T_h\} \subset L(X)$ be asymptotically equivalent.*

- i) *If $\{S_h\}$ is a bounded family of operators, then $\{T_h\}$ is also bounded and conversely;*
- ii) *$\{S_h\}, \{T_h\}$ are asymptotically commuting (i.e. $\lim_{h \rightarrow 0} \|S_h T_h - T_h S_h\| = 0$);*
- iii) *Let $\{U_h\} \subset L(X)$ be a bounded family of operators such that $\lim_{h \rightarrow 0} \|S_h U_h - U_h S_h\| = 0$. Then $\lim_{h \rightarrow 0} \|U_h T_h - T_h U_h\| = 0$.*

Proof. i) If $\{S_h\}$ is a bounded family of operators, then there is $\limsup_{h \rightarrow 0} \|S_h\| < \infty$. Since

$$\lim_{h \rightarrow 0} \|S_h - T_h\| = 0,$$

it follows that

$$\begin{aligned} \limsup_{h \rightarrow 0} \|T_h\| &= \limsup_{h \rightarrow 0} \|T_h - S_h + S_h\| \\ &\leq \lim_{h \rightarrow 0} \|S_h - T_h\| + \limsup_{h \rightarrow 0} \|S_h\| < \infty. \end{aligned}$$

Therefore $\{T_h\}$ is a bounded family of operators.

Analogously we can prove that if $\{T_h\}$ is a bounded family of operators, then $\{S_h\}$ is a bounded family of operators.

ii)

$$\begin{aligned} \limsup_{h \rightarrow 0} \|S_h T_h - T_h S_h\| &= \limsup_{h \rightarrow 0} \|S_h T_h - S_h^2 + S_h^2 - T_h S_h\| \leq \\ &\leq \limsup_{h \rightarrow 0} \|S_h (S_h - T_h)\| + \limsup_{h \rightarrow 0} \|(S_h - T_h) S_h\| \\ &\leq 2 \limsup_{h \rightarrow 0} \|S_h\| \|S_h - T_h\| \leq 0. \end{aligned}$$

iii)

$$\begin{aligned} \limsup_{h \rightarrow 0} \|T_h U_h - U_h T_h\| &= \limsup_{h \rightarrow 0} \|T_h U_h - S_h U_h + S_h U_h - U_h S_h + U_h S_h - U_h T_h\| \leq \\ &\leq \limsup_{h \rightarrow 0} \|T_h U_h - S_h U_h\| + \limsup_{h \rightarrow 0} \|S_h U_h - U_h S_h\| + \limsup_{h \rightarrow 0} \|U_h S_h - U_h T_h\| \leq \\ &\leq 2 \limsup_{h \rightarrow 0} \|U_h\| \|T_h - S_h\| \end{aligned}$$

Since $\{U_h\}$ is a bounded family of operators, then there is $\limsup_{h \rightarrow 0} \|U_h\| < \infty$.
So

$$\lim_{h \rightarrow 0} \|U_h T_h - T_h U_h\| = 0.$$

□

Proposition 8. Let $\{S_h\}, \{T_h\} \subset L(X)$ be two bounded families of operators such that $\lim_{h \rightarrow 0} \|S_h T_h - T_h S_h\| = 0$. Then

- i) $\lim_{h \rightarrow 0} \|S_h^n T_h^m - T_h^m S_h^n\| = 0$, for any $n, m \in \mathbb{N}$;
- ii) $\lim_{h \rightarrow 0} \|(S_h - T_h)^{[n]}\| = \lim_{h \rightarrow 0} \|(S_h - T_h)^n\|$, for any $n \in \mathbb{N}$;
- iii) $\lim_{h \rightarrow 0} \|(S_h T_h)^n - S_h^n T_h^n\| = 0$, for any $n \in \mathbb{N}$.

Proof. i) We prove that $\lim_{h \rightarrow 0} \|S_h^n T_h - T_h S_h^n\| = 0$, for any $n \in \mathbb{N}$. For $n = 2$ we have

$$\begin{aligned} \limsup_{h \rightarrow 0} \|S_h^2 T_h - T_h S_h^2\| &= \limsup_{h \rightarrow 0} \|S_h (S_h T_h) - S_h (T_h S_h) + (S_h T_h) S_h - (T_h S_h) S_h\| \leq \\ &\leq 2 \limsup_{h \rightarrow 0} \|(S_h T_h) - (T_h S_h)\| \|S_h\| = 0. \end{aligned}$$

For $n = 3$ we have

$$\begin{aligned} \limsup_{h \rightarrow 0} \|S_h^3 T_h - T_h S_h^3\| &= \limsup_{h \rightarrow 0} \|S_h (S_h^2 T_h) - S_h (T_h S_h^2) + (S_h T_h) S_h^2 - (T_h S_h) S_h^2\| \leq \\ &\leq \limsup_{h \rightarrow 0} \|S_h^2 T_h - T_h S_h^2\| \|S_h\| + \lim_{h \rightarrow 0} \|S_h T_h - T_h S_h\| \|S_h^2\| = 0. \end{aligned}$$

Considering relation $\lim_{h \rightarrow 0} \|S_h^n T_h - T_h S_h^n\| = 0$ true we prove that

$$\lim_{h \rightarrow 0} \|S_h^{n+1} T_h - T_h S_h^{n+1}\| = 0.$$

$$\begin{aligned} \limsup_{h \rightarrow 0} \|S_h^{n+1} T_h - T_h S_h^{n+1}\| &= \limsup_{h \rightarrow 0} \|S_h (S_h^n T_h) - S_h (T_h S_h^n) + (S_h T_h) S_h^n - (T_h S_h) S_h^n\| \leq \\ &\leq \limsup_{h \rightarrow 0} \|S_h^n T_h - T_h S_h^n\| \|S_h\| + \limsup_{h \rightarrow 0} \|S_h T_h - T_h S_h\| \|S_h^n\| = 0. \end{aligned}$$

Applying above relation to S_h^n and T_h , it follows that

$$\lim_{h \rightarrow 0} \|S_h^n T_h^m - T_h^m S_h^n\| = 0,$$

for every $n, m \in \mathbb{N}$.

ii) and iii) can be proved analogously i). □

Proposition 9. Let $\{S_h\}, \{T_h\} \subset L(X)$ be two bounded families of operators.

i) If $\{S_h\}, \{T_h\}$ are asymptotically equivalent, then are asymptotically quasinilpotent equivalent.

ii) If $\lim_{h \rightarrow 0} \|S_h T_h - T_h S_h\| = 0$ and $\lim_{n \rightarrow \infty} \limsup_{h \rightarrow 0} \|(S_h - T_h)^{[n]}\|^{\frac{1}{n}} = 0$, then $\{S_h\}, \{T_h\}$ are asymptotically quasinilpotent equivalent, i.e.

$$\lim_{n \rightarrow \infty} \limsup_{h \rightarrow 0} \|(T_h - S_h)^{[n]}\|^{\frac{1}{n}} = 0.$$

iii) Let $\{A_h\} \subset L(X)$ be a bounded families of operators. If $\{S_h\}, \{T_h\}$ are asymptotically quasinilpotent equivalent and $\lim_{h \rightarrow 0} \|S_h A_h - A_h S_h\| = 0$, then it is not necessary that $\lim_{h \rightarrow 0} \|T_h A_h - A_h T_h\| = 0$.

Proof. i) We prove that

$$\lim_{h \rightarrow 0} \|(S_h - T_h)^{[n]}\| = \lim_{h \rightarrow 0} \|(T_h - S_h)^{[n]}\| = 0,$$

for any $n \in \mathbb{N}$.

Since $(T - S)^{[n+1]} = T(T - S)^{[n]} - (T - S)^{[n]}S$, for any $n \in \mathbb{N}$, taking $n = 2$, it follows that

$$\begin{aligned} \limsup_{h \rightarrow 0} \|(T_h - S_h)^{[2]}\| &= \limsup_{h \rightarrow 0} \|T_h(T_h - S_h) - (T_h - S_h)S_h\| \leq \\ &\leq \limsup_{h \rightarrow 0} \|T_h(T_h - S_h)\| + \limsup_{h \rightarrow 0} \|(T_h - S_h)S_h\| \leq \\ &\leq \limsup_{h \rightarrow 0} \|T_h\| \|(T_h - S_h)\| + \limsup_{h \rightarrow 0} \|(T_h - S_h)\| \|S_h\| \leq 0. \end{aligned}$$

By induction, we prove that if $\lim_{h \rightarrow 0} \|(T_h - S_h)^{[n]}\| = 0$, then $\lim_{h \rightarrow 0} \|(T_h - S_h)^{[n+1]}\| = 0$.

$$\begin{aligned} \limsup_{h \rightarrow 0} \|(T_h - S_h)^{[n+1]}\| &= \limsup_{h \rightarrow 0} \|T_h(T_h - S_h)^{[n]} - (T_h - S_h)^{[n]}S_h\| \leq \\ &\leq \limsup_{h \rightarrow 0} \|T_h(T_h - S_h)^{[n]}\| + \limsup_{h \rightarrow 0} \|(T_h - S_h)^{[n]}S_h\| \leq \\ &\leq \limsup_{h \rightarrow 0} \|T_h\| \|(T_h - S_h)^{[n]}\| + \limsup_{h \rightarrow 0} \|(T_h - S_h)^{[n]}\| \|S_h\| \leq 0. \end{aligned}$$

Similarly we can show that $\lim_{h \rightarrow 0} \|(S_h - T_h)^{[n]}\| = 0$, for any $n \in \mathbb{N}$.

When $n \rightarrow \infty$, we obtain

$$\lim_{n \rightarrow \infty} \limsup_{h \rightarrow 0} \|(S_h - T_h)^{[n]}\|^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \limsup_{h \rightarrow 0} \|(T_h - S_h)^{[n]}\|^{\frac{1}{n}} = 0.$$

ii) We remember that for any two bounded linear operators T and S , we have

$$(T - S)^{[n]} = \sum_{k=0}^n (-1)^{n-k} C_k^n T^k S^{n-k} = (S - T)^{[n]} + \sum_{k=0}^{n-1} (-1)^{n-1-k} C_k^n (T^k S^{n-k} - S^{n-k} T^k),$$

where $n \in \mathbb{N}$.

Applying above relation to S_h și T_h , when $h \rightarrow 0$, we obtain

$$\begin{aligned} \limsup_{h \rightarrow 0} \|(T_h - S_h)^{[n]}\| &= \\ &= \limsup_{h \rightarrow 0} \left\| (S_h - T_h)^{[n]} - \sum_{k=0}^{n-1} (-1)^{n-1-k} C_k^n (T_h^k S_h^{n-k} - S_h^{n-k} T_h^k) \right\| \leq \end{aligned}$$

$$\begin{aligned}
&\leq \limsup_{h \rightarrow 0} \left\| (S_h - T_h)^{[n]} \right\| + \limsup_{h \rightarrow 0} \left\| \sum_{k=0}^{n-1} (-1)^{n-1-k} C_k^n (T_h^k S_h^{n-k} - S_h^{n-k} T_h^k) \right\| \leq \\
&\leq \limsup_{h \rightarrow 0} \left\| (S_h - T_h)^{[n]} \right\| + \sum_{k=0}^{n-1} C_k^n \limsup_{h \rightarrow 0} \left\| T_h^k S_h^{n-k} - S_h^{n-k} T_h^k \right\|.
\end{aligned}$$

By Proposition 8 ii), it follows

$$\limsup_{h \rightarrow 0} \left\| (T_h - S_h)^{[n]} \right\| \leq \limsup_{h \rightarrow 0} \left\| (S_h - T_h)^{[n]} \right\|,$$

for any $n \in \mathbb{N}$.

Analogously we can prove that $\limsup_{h \rightarrow 0} \left\| (S_h - T_h)^{[n]} \right\| \leq \limsup_{h \rightarrow 0} \left\| (T_h - S_h)^{[n]} \right\|$.

iii) We suppose that the relation $\lim_{h \rightarrow 0} \|T_h A_h - A_h T_h\| = 0$ is true. Then, taking $A_h = S_h$, for any $h \in (0, 1]$, since

$$\lim_{h \rightarrow 0} \|S_h^2 - S_h^2\| = 0,$$

it follows

$$\lim_{h \rightarrow 0} \|S_h T_h - T_h S_h\| = 0,$$

fact that is not true. □

Proposition 10. *Let $\{S_h\}, \{T_h\} \subset L(X)$ be two asymptotically quasinilpotent equivalent families and $\{A_h\} \subset L(X)$ a bounded family. Then*

- i) *The families $\{S_h + A_h\}, \{T_h + A_h\}$ are asymptotically quasinilpotent equivalent;*
- ii) *If $\{A_h\} \subset L(X)$ is a bounded family such that $\lim_{h \rightarrow 0} \|S_h A_h - A_h S_h\| = 0$ and $\lim_{h \rightarrow 0} \|T_h A_h - A_h T_h\| = 0$, the families $\{S_h A_h\}, \{T_h A_h\}$ are asymptotically quasinilpotent equivalent.*

Proof. i) Since $\{S_h\}, \{T_h\}$ are asymptotically quasinilpotent equivalent, i.e.

$$\lim_{n \rightarrow \infty} \limsup_{h \rightarrow 0} \left\| (T_h - S_h)^{[n]} \right\|^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \limsup_{h \rightarrow 0} \left\| (S_h - T_h)^{[n]} \right\|^{\frac{1}{n}} = 0,$$

then

$$\begin{aligned}
&\lim_{n \rightarrow \infty} \limsup_{h \rightarrow 0} \left\| ((T_h + A_h) - (S_h + A_h))^{[n]} \right\|^{\frac{1}{n}} \\
&= \lim_{n \rightarrow \infty} \limsup_{h \rightarrow 0} \left\| ((S_h + A_h) - (T_h + A_h))^{[n]} \right\|^{\frac{1}{n}} \\
&= 0,
\end{aligned}$$

so $\{S_h + A_h\}$, $\{T_h + A_h\}$ are asymptotically quasinilpotent equivalent.

ii) Since $\lim_{h \rightarrow 0} \|S_h A_h - A_h S_h\| = 0$ and $\lim_{h \rightarrow 0} \|T_h A_h - A_h T_h\| = 0$, taking into account Proposition 9, it follows

$$\begin{aligned}
& \limsup_{h \rightarrow 0} \left\| (T_h A_h - S_h A_h)^{[n]} - (T_h - S_h)^{[n]} A_h^n \right\| = \\
& = \limsup_{h \rightarrow 0} \left\| \sum_{k=0}^n (-1)^{n-k} C_n^k (T_h A_h)^k (S_h A_h)^{n-k} - \sum_{k=0}^n (-1)^{n-k} C_n^k T_h^k S_h^{n-k} A_h^n \right\| = \\
& = \limsup_{h \rightarrow 0} \left\| \sum_{k=0}^n (-1)^{n-k} C_n^k ((T_h A_h)^k (S_h A_h)^{n-k} - T_h^k S_h^{n-k} A_h^k A_h^{n-k}) \right\| \leq \\
& \leq \sum_{k=0}^n C_n^k \limsup_{h \rightarrow 0} \| (T_h A_h)^k (S_h A_h)^{n-k} - T_h^k A_h^k S_h^{n-k} A_h^{n-k} + \\
& \quad + T_h^k A_h^k S_h^{n-k} A_h^{n-k} - T_h^k S_h^{n-k} A_h^k A_h^{n-k} \| \leq \\
& \leq \sum_{k=0}^n C_n^k \limsup_{h \rightarrow 0} \| (T_h A_h)^k (S_h A_h)^{n-k} - T_h^k A_h^k S_h^{n-k} A_h^{n-k} \| + \\
& + \sum_{k=0}^n C_n^k \limsup_{h \rightarrow 0} \| T_h^k A_h^k S_h^{n-k} A_h^{n-k} - T_h^k S_h^{n-k} A_h^k A_h^{n-k} \| \leq \\
& \leq \sum_{k=0}^n C_n^k \limsup_{h \rightarrow 0} \| (T_h A_h)^k (S_h A_h)^{n-k} - (T_h A_h)^k S_h^{n-k} A_h^{n-k} + \\
& \quad + (T_h A_h)^k S_h^{n-k} A_h^{n-k} - T_h^k A_h^k S_h^{n-k} A_h^{n-k} \| + \\
& + \sum_{k=0}^n C_n^k \limsup_{h \rightarrow 0} \| T_h^k \| \| A_h^k S_h^{n-k} - S_h^{n-k} A_h^k \| \| A_h^{n-k} \| \leq \\
& \leq \sum_{k=0}^n C_n^k \limsup_{h \rightarrow 0} \| (T_h A_h)^k (S_h A_h)^{n-k} - (T_h A_h)^k S_h^{n-k} A_h^{n-k} \| + \\
& + \sum_{k=0}^n C_n^k \limsup_{h \rightarrow 0} \| (T_h A_h)^k S_h^{n-k} A_h^{n-k} - T_h^k A_h^k S_h^{n-k} A_h^{n-k} \| \leq \\
& \leq \sum_{k=0}^n C_n^k \limsup_{h \rightarrow 0} \| (T_h A_h)^k \| \| (S_h A_h)^{n-k} - S_h^{n-k} A_h^{n-k} \| + \\
& + \sum_{k=0}^n C_n^k \limsup_{h \rightarrow 0} \| (T_h A_h)^k - T_h^k A_h^k S_h^{n-k} A_h^{n-k} \| \| S_h^{n-k} \| \| A_h^{n-k} \| = 0.
\end{aligned}$$

Having in view that $\{S_h\}$, $\{T_h\}$ are asymptotically quasinilpotent equivalent and taking into account the above relation, it results

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \limsup_{h \rightarrow 0} \left\| (T_h A_h - S_h A_h)^{[n]} \right\|^{\frac{1}{n}} \\
&= \lim_{n \rightarrow \infty} \limsup_{h \rightarrow 0} \left\| (T_h - S_h)^{[n]} A_h^n \right\|^{\frac{1}{n}} \\
&\leq \lim_{n \rightarrow \infty} \left(\limsup_{h \rightarrow 0} \left\| (T_h - S_h)^{[n]} \right\| \|A_h^n\| \right)^{\frac{1}{n}} \\
&\leq \lim_{n \rightarrow \infty} \lim_{h \rightarrow 0} \left\| (T_h - S_h)^{[n]} \right\|^{\frac{1}{n}} \limsup_{h \rightarrow 0} \|A_h\| = 0.
\end{aligned}$$

Analogously we can prove that $\lim_{n \rightarrow \infty} \limsup_{h \rightarrow 0} \left\| (S_h A_h - T_h A_h)^{[n]} \right\|^{\frac{1}{n}} = 0$. \square

3 Spectrum of a family of operators

Let be the sets

$$\begin{aligned}
C_b((0, 1], B(X)) &= \{ \varphi : (0, 1] \rightarrow B(X) \mid \varphi(h) = T_h \text{ such that } \varphi \text{ is continuous and bounded} \} = \\
&= \left\{ \{T_h\}_{h \in (0, 1]} \subset B(X) \mid \{T_h\}_{h \in (0, 1]} \text{ is a bounded family, i.e. } \sup_{h \in (0, 1]} \|T_h\| < \infty \right\}.
\end{aligned}$$

and

$$\begin{aligned}
C_0((0, 1], B(X)) &= \left\{ \varphi \in C_b((0, 1], B(X)) \mid \lim_{h \rightarrow 0} \|\varphi(h)\| = 0 \right\} = \\
&= \left\{ \{T_h\}_{h \in (0, 1]} \subset B(X) \mid \lim_{h \rightarrow 0} \|T_h\| = 0 \right\}.
\end{aligned}$$

$C_b((0, 1], B(X))$ is a Banach algebra non-commutative with norm

$$\|\{T_h\}\| = \sup_{h \in (0, 1]} \|T_h\|,$$

and $C_0((0, 1], B(X))$ is a closed bilateral ideal of $C_b((0, 1], B(X))$. Therefore the quotient algebra $C_b((0, 1], B(X)) / C_0((0, 1], B(X))$, which will be called from now B_∞ , is also a Banach algebra with quotient norm

$$\|\{\dot{T}_h\}\| = \inf_{\{U_h\}_{h \in (0, 1]} \in C_0((0, 1], B(X))} \|\{T_h\} + \{U_h\}\| = \inf_{\{S_h\}_{h \in (0, 1]} \in \{\dot{T}_h\}} \|\{S_h\}\|.$$

Then

$$\|\{\dot{T}_h\}\| = \inf_{\{S_h\}_{h \in (0, 1]} \in \{\dot{T}_h\}} \|\{S_h\}\| \leq \|\{S_h\}\| = \sup_{h \in (0, 1]} \|S_h\|,$$

for any $\{S_h\}_{h \in (0, 1]} \in \{\dot{T}_h\}$. Moreover,

$$\left\| \{\dot{T}_h\} \right\| = \inf_{\{S_h\}_{h \in (0,1]} \in \{\dot{T}_h\}} \|\{S_h\}\| = \inf_{\{S_h\}_{h \in (0,1]} \in \{\dot{T}_h\}} \sup_{h \in (0,1]} \|S_h\|.$$

If two bounded families $\{T_h\}_{h \in (0,1]}$, $\{S_h\}_{h \in (0,1]} \subset B(X)$ are asymptotically equivalent, then $\lim_{h \rightarrow 0} \|S_h - T_h\| = 0$, i.e. $\{T_h - S_h\}_{h \in (0,1]} \in C_0((0,1], B(X))$.

Let $\{T_h\}_{h \in (0,1]}$, $\{S_h\}_{h \in (0,1]} \in C_b((0,1], B(X))$ be asymptotically equivalent. Then

$$\limsup_{h \rightarrow 0} \|S_h\| = \limsup_{h \rightarrow 0} \|T_h\|.$$

Since

$$\limsup_{h \rightarrow 0} \|S_h\| \leq \sup_{h \in (0,1]} \|S_h\|,$$

results that

$$\limsup_{h \rightarrow 0} \|S_h\| = \inf_{\{S_h\}_{h \in (0,1]} \in \{\dot{T}_h\}} \limsup_{h \rightarrow 0} \|S_h\| \leq \inf_{\{S_h\}_{h \in (0,1]} \in \{\dot{T}_h\}} \sup_{h \in (0,1]} \|S_h\| = \left\| \{\dot{T}_h\} \right\|,$$

for any $\{S_h\}_{h \in (0,1]} \in \{\dot{T}_h\}$.

In particular

$$\lim_{h \rightarrow 0} \lim_{h \rightarrow 0} \|T_h\| \leq \left\| \{\dot{T}_h\} \right\| \leq \|\{T_h\}\| = \sup_{h \in (0,1]} \|T_h\|.$$

Definition 11. We call the resolvent set of a family of operators $\{S_h\} \in C_b((0,1], B(X))$ the set

$$\begin{aligned} r(\{S_h\}) &= \{ \lambda \in \mathbb{C} \mid \exists \{\mathcal{R}(\lambda, S_h)\} \in C_b((0,1], B(X)), \lim_{h \rightarrow 0} \|(\lambda I - S_h) \mathcal{R}(\lambda, S_h) - I\| = \\ &= \lim_{h \rightarrow 0} \|\mathcal{R}(\lambda, S_h) (\lambda I - S_h) - I\| = 0 \} \end{aligned}$$

We call the spectrum of a family of operators $\{S_h\} \in C_b((0,1], B(X))$ the set

$$Sp(\{S_h\}) = \mathbb{C} \setminus r(\{S_h\}).$$

Remark 12. i) If $\lambda \in r(S_h)$ for any $h \in (0,1]$, then $\lambda \in r(\{S_h\})$. Therefore

$$\bigcap_{h \in (0,1]} r(S_h) \subseteq r(\{S_h\});$$

ii) If $\lambda \in Sp(\{S_h\})$, then $|\lambda| \leq \limsup_{h \rightarrow 0} \|S_h\|$;

iii) If $\|S_h\| < |\lambda|$ for any $h \in (0,1]$, then $\lambda \in r(\{S_h\})$;

iv) $r(\{S_h\})$ is an open set of \mathbb{C} and $Sp(\{S_h\})$ is a compact set of \mathbb{C} .

Proof. iv) Let $\lambda \in r(\{S_h\})$. From Definition 11, it follows that there is $\{\mathcal{R}(\lambda, S_h)\} \in C_b((0,1], B(X))$ such that

$$\lim_{h \rightarrow 0} \|(\lambda I - S_h) \mathcal{R}(\lambda, S_h) - I\| = \lim_{h \rightarrow 0} \|\mathcal{R}(\lambda, S_h) (\lambda I - S_h) - I\| = 0.$$

Let $\mu \in D(\lambda, \frac{1}{\limsup_{h \rightarrow 0} \|\mathcal{R}(\lambda, S_h)\|})$. So

$$|\lambda - \mu| < \frac{1}{\limsup_{h \rightarrow 0} \|\mathcal{R}(\lambda, S_h)\|}.$$

According to ii), it follows $1 \in r(\{(\lambda - \mu)\mathcal{R}(\lambda, S_h)\})$, therefore there is $\{\mathcal{R}(1, (\lambda - \mu)\mathcal{R}(\lambda, S_h))\} \in C_b((0, 1], B(X))$ such that

$$\begin{aligned} & \lim_{h \rightarrow 0} \|(I - (\lambda - \mu)\mathcal{R}(\lambda, S_h))\mathcal{R}(1, (\lambda - \mu)\mathcal{R}(\lambda, S_h)) - I\| = \\ & = \lim_{h \rightarrow 0} \|\mathcal{R}(1, (\lambda - \mu)\mathcal{R}(\lambda, S_h))(I - (\lambda - \mu)\mathcal{R}(\lambda, S_h)) - I\| = 0. \end{aligned}$$

Having in view the above relation, it results

$$\begin{aligned} & \limsup_{h \rightarrow 0} \|(\mu I - S_h)\mathcal{R}(\lambda, S_h)\mathcal{R}(1, (\lambda - \mu)\mathcal{R}(\lambda, S_h)) - I\| = \\ & = \limsup_{h \rightarrow 0} \|(\lambda I - S_h)\mathcal{R}(\lambda, S_h)\mathcal{R}(1, (\lambda - \mu)\mathcal{R}(\lambda, S_h)) - \\ & \quad - (\lambda - \mu)\mathcal{R}(\lambda, S_h)\mathcal{R}(1, (\lambda - \mu)\mathcal{R}(\lambda, S_h)) - I\| = \\ & = \limsup_{h \rightarrow 0} \|((\lambda I - S_h)\mathcal{R}(\lambda, S_h) - I)\mathcal{R}(1, (\lambda - \mu)\mathcal{R}(\lambda, S_h)) + \mathcal{R}(1, (\lambda - \mu)\mathcal{R}(\lambda, S_h)) - \\ & \quad - (\lambda - \mu)\mathcal{R}(\lambda, S_h)\mathcal{R}(1, (\lambda - \mu)\mathcal{R}(\lambda, S_h)) - I\| \leq \\ & \leq \limsup_{h \rightarrow 0} \|((\lambda I - S_h)\mathcal{R}(\lambda, S_h) - I)\mathcal{R}(1, (\lambda - \mu)\mathcal{R}(\lambda, S_h))\| + \\ & + \limsup_{h \rightarrow 0} \|\mathcal{R}(1, (\lambda - \mu)\mathcal{R}(\lambda, S_h)) - (\lambda - \mu)\mathcal{R}(\lambda, S_h)\mathcal{R}(1, (\lambda - \mu)\mathcal{R}(\lambda, S_h)) - I\| \leq \\ & \leq \limsup_{h \rightarrow 0} \|((\lambda I - S_h)\mathcal{R}(\lambda, S_h) - I)\| \|\mathcal{R}(1, (\lambda - \mu)\mathcal{R}(\lambda, S_h))\| + \\ & + \lim_{h \rightarrow 0} \|(I - (\lambda - \mu)\mathcal{R}(\lambda, S_h))\mathcal{R}(1, (\lambda - \mu)\mathcal{R}(\lambda, S_h)) - I\| = 0, \end{aligned}$$

so $\mu \in r(\{S_h\})$, for every $\mu \in D(\lambda, \frac{1}{\limsup_{h \rightarrow 0} \|\mathcal{R}(\lambda, S_h)\|})$. Therefore, for any $\lambda \in r(\{S_h\})$, there is an open disk $D(\lambda, \frac{1}{\limsup_{h \rightarrow 0} \|\mathcal{R}(\lambda, S_h)\|})$ such that $D(\lambda, \frac{1}{\limsup_{h \rightarrow 0} \|\mathcal{R}(\lambda, S_h)\|}) \subset r(\{S_h\})$.

If $\{S_h\}$ is a bounded family, from ii) we have

$$|\lambda| \leq \limsup_{h \rightarrow 0} \|S_h\| < \infty,$$

for any $\lambda \in Sp(\{S_h\})$, so $Sp(\{S_h\})$ is a compact set. \square

Proposition 13. Let $\{S_h\} \in C_b((0, 1], B(X))$ be a family of operators and $\lambda \in r(\{S_h\})$. Then, for any $\{\mathcal{R}(\lambda, S_h)\} \in C_b((0, 1], B(X))$ such that

$$\lim_{h \rightarrow 0} \|(\lambda I - S_h) \mathcal{R}(\lambda, S_h) - I\| = \lim_{h \rightarrow 0} \|\mathcal{R}(\lambda, S_h) (\lambda I - S_h) - I\| = 0,$$

we have

$$\lim_{h \rightarrow 0} \|S_h \mathcal{R}(\lambda, S_h) - \mathcal{R}(\lambda, S_h) S_h\| = 0.$$

Proof. Let $\lambda \in r(\{S_h\})$ and $\{\mathcal{R}(\lambda, S_h)\} \in C_b((0, 1], B(X))$ such that

$$\lim_{h \rightarrow 0} \|(\lambda I - S_h) \mathcal{R}(\lambda, S_h) - I\| = \lim_{h \rightarrow 0} \|\mathcal{R}(\lambda, S_h) (\lambda I - S_h) - I\| = 0.$$

Using this relation we have

$$\begin{aligned} \limsup_{h \rightarrow 0} \|S_h \mathcal{R}(\lambda, S_h) - \mathcal{R}(\lambda, S_h) S_h\| &= \limsup_{h \rightarrow 0} \|\mathcal{R}(\lambda, S_h) (\lambda I - S_h) - (\lambda I - S_h) \mathcal{R}(\lambda, S_h)\| = \\ &= \limsup_{h \rightarrow 0} \|\mathcal{R}(\lambda, S_h) (\lambda I - S_h) - I + I - (\lambda I - S_h) \mathcal{R}(\lambda, S_h)\| \leq \\ &\leq \lim_{h \rightarrow 0} \|(\lambda I - S_h) \mathcal{R}(\lambda, S_h) - I\| + \lim_{h \rightarrow 0} \|\mathcal{R}(\lambda, S_h) (\lambda I - S_h) - I\| = 0. \end{aligned}$$

□

Proposition 14. (resolvent equation - asymptotic) Let $\{S_h\} \in C_b((0, 1], B(X))$ be a bounded family and $\lambda, \mu \in r(\{S_h\})$. Then

$$\lim_{h \rightarrow 0} \|\mathcal{R}(\lambda, S_h) - \mathcal{R}(\mu, S_h) - (\mu - \lambda) \mathcal{R}(\lambda, S_h) \mathcal{R}(\mu, S_h)\| = 0.$$

Proof. Since $\{\mathcal{R}(\lambda, S_h)\}$ and $\{\mathcal{R}(\mu, S_h)\}$ are bounded, we have

$$\begin{aligned} \limsup_{h \rightarrow 0} \|\mathcal{R}(\lambda, S_h) - \mathcal{R}(\mu, S_h) - (\mu - \lambda) \mathcal{R}(\lambda, S_h) \mathcal{R}(\mu, S_h)\| &= \\ &= \limsup_{h \rightarrow 0} \|\mathcal{R}(\lambda, S_h) (I - \mu \mathcal{R}(\mu, S_h)) - (I - \lambda \mathcal{R}(\lambda, S_h)) \mathcal{R}(\mu, S_h)\| = \\ &= \limsup_{h \rightarrow 0} \|\mathcal{R}(\lambda, S_h) (I - (\mu I - S_h) \mathcal{R}(\mu, S_h)) - (I - \mathcal{R}(\lambda, S_h) (\lambda I - S_h)) \mathcal{R}(\mu, S_h)\| \leq \\ &\leq \limsup_{h \rightarrow 0} \|\mathcal{R}(\lambda, S_h) (I - (\mu I - S_h) \mathcal{R}(\mu, S_h))\| + \limsup_{h \rightarrow 0} \|(I - \mathcal{R}(\lambda, S_h) (\lambda I - S_h)) \mathcal{R}(\mu, S_h)\| \leq \\ &\leq \limsup_{h \rightarrow 0} \|\mathcal{R}(\lambda, S_h)\| \|I - (\mu I - S_h) \mathcal{R}(\mu, S_h)\| + \\ &+ \limsup_{h \rightarrow 0} \|I - \mathcal{R}(\lambda, S_h) (\lambda I - S_h)\| \|\mathcal{R}(\mu, S_h)\| \leq 0. \end{aligned}$$

□

Corollary 15. Let $\{S_h\} \in C_b((0, 1], B(X))$ be a bounded family and $\lambda, \mu \in r(\{S_h\})$ be not-equal. Then

$$\lim_{h \rightarrow 0} \|\mathcal{R}(\lambda, S_h) \mathcal{R}(\mu, S_h) - \mathcal{R}(\mu, S_h) \mathcal{R}(\lambda, S_h)\| = 0.$$

Proof.

$$\begin{aligned}
& \limsup_{h \rightarrow 0} \|\mathcal{R}(\lambda, S_h) \mathcal{R}(\mu, S_h) - \mathcal{R}(\mu, S_h) \mathcal{R}(\lambda, S_h)\| = \\
& = \frac{1}{|\lambda - \mu|} \limsup_{h \rightarrow 0} \|(\lambda - \mu) \mathcal{R}(\lambda, S_h) \mathcal{R}(\mu, S_h) + (\mu - \lambda) \mathcal{R}(\mu, S_h) \mathcal{R}(\lambda, S_h)\| = \\
& = \frac{1}{|\lambda - \mu|} \limsup_{h \rightarrow 0} \|[\mathcal{R}(\lambda, S_h) - \mathcal{R}(\mu, S_h) - (\mu - \lambda) \mathcal{R}(\lambda, S_h) \mathcal{R}(\mu, S_h)] + \\
& \quad + [\mathcal{R}(\mu, S_h) - \mathcal{R}(\lambda, S_h) - (\lambda - \mu) \mathcal{R}(\mu, S_h) \mathcal{R}(\lambda, S_h)]\| \leq \\
& \leq \frac{1}{|\lambda - \mu|} \lim_{h \rightarrow 0} \|[\mathcal{R}(\lambda, S_h) - \mathcal{R}(\mu, S_h) - (\mu - \lambda) \mathcal{R}(\lambda, S_h) \mathcal{R}(\mu, S_h)]\| + \\
& \quad + \frac{1}{|\lambda - \mu|} \lim_{h \rightarrow 0} \|[\mathcal{R}(\mu, S_h) - \mathcal{R}(\lambda, S_h) - (\lambda - \mu) \mathcal{R}(\mu, S_h) \mathcal{R}(\lambda, S_h)]\| = 0.
\end{aligned}$$

□

Proposition 16. Let $\{S_h\} \in C_b((0, 1], B(X))$ be a bounded family. If $\lambda \in r(\{S_h\})$ and $\{\mathcal{R}_i(\lambda, S_h)\} \in C_b((0, 1], B(X))$, $i = \overline{1, 2}$ such that

$$\lim_{h \rightarrow 0} \|(\lambda I - S_h) \mathcal{R}_i(\lambda, S_h) - I\| = \lim_{h \rightarrow 0} \|\mathcal{R}_i(\lambda, S_h) (\lambda I - S_h) - I\| = 0$$

for $i = \overline{1, 2}$, then

$$\lim_{h \rightarrow 0} \|\mathcal{R}_1(\lambda, S_h) - \mathcal{R}_2(\lambda, S_h)\| = 0.$$

Proof. Let $\lambda \in r(\{S_h\})$ and $\{\mathcal{R}_i(\lambda, S_h)\} \in C_b((0, 1], B(X))$, $i = \overline{1, 2}$, such that

$$\lim_{h \rightarrow 0} \|(\lambda I - S_h) \mathcal{R}_i(\lambda, S_h) - I\| = \lim_{h \rightarrow 0} \|\mathcal{R}_i(\lambda, S_h) (\lambda I - S_h) - I\| = 0$$

Therefore

$$\begin{aligned}
& \limsup_{h \rightarrow 0} \|\mathcal{R}_1(\lambda, S_h) - \mathcal{R}_2(\lambda, S_h)\| = \\
& = \limsup_{h \rightarrow 0} \|\mathcal{R}_1(\lambda, S_h) - \mathcal{R}_2(\lambda, S_h) - (\lambda - \lambda) \mathcal{R}_1(\lambda, S_h) \mathcal{R}_2(\lambda, S_h)\| = \\
& = \limsup_{h \rightarrow 0} \|\mathcal{R}_1(\lambda, S_h) (I - \lambda \mathcal{R}_2(\lambda, S_h)) - (I - \lambda \mathcal{R}_1(\lambda, S_h)) \mathcal{R}_2(\lambda, S_h)\| = \\
& = \limsup_{h \rightarrow 0} \|\mathcal{R}_1(\lambda, S_h) (I - (\lambda I - S_h) \mathcal{R}_2(\lambda, S_h)) - (I - \mathcal{R}_1(\lambda, S_h) (\lambda I - S_h)) \mathcal{R}_2(\lambda, S_h)\| \leq \\
& \leq \limsup_{h \rightarrow 0} \|\mathcal{R}_1(\lambda, S_h) (I - (\lambda I - S_h) \mathcal{R}_2(\lambda, S_h))\| + \\
& \quad + \limsup_{h \rightarrow 0} \|(I - \mathcal{R}_1(\lambda, S_h) (\lambda I - S_h)) \mathcal{R}_2(\lambda, S_h)\| \leq \\
& \leq \limsup_{h \rightarrow 0} \|\mathcal{R}_1(\lambda, S_h)\| \|I - (\lambda I - S_h) \mathcal{R}_2(\lambda, S_h)\| + \\
& \quad + \limsup_{h \rightarrow 0} \|I - \mathcal{R}_1(\lambda, S_h) (\lambda I - S_h)\| \|\mathcal{R}_2(\lambda, S_h)\| \leq 0
\end{aligned}$$

□

Proposition 17. Let $\{S_h\} \in C_b((0, 1], B(X))$ be a bounded family, $\lambda \in r(\{S_h\})$ and $\{\mathcal{R}(\lambda, S_h)\} \in C_b((0, 1], B(X))$ such that

$$\lim_{h \rightarrow 0} \|(\lambda I - S_h) \mathcal{R}(\lambda, S_h) - I\| = \lim_{h \rightarrow 0} \|\mathcal{R}(\lambda, S_h)(\lambda I - S_h) - I\| = 0.$$

If $\{R_h\} \in C_b((0, 1], B(X))$ is a bounded family such that it is asymptotically equivalent with $\{\mathcal{R}(\lambda, S_h)\} \in C_b((0, 1], B(X))$, then

$$\lim_{h \rightarrow 0} \|(\lambda I - S_h) R_h - I\| = \lim_{h \rightarrow 0} \|R_h(\lambda I - S_h) - I\| = 0.$$

Proof. Let $\lambda \in r(\{S_h\})$. It results

$$\begin{aligned} & \limsup_{h \rightarrow 0} \|(\lambda I - S_h) R_h - I\| = \\ &= \limsup_{h \rightarrow 0} \|(\lambda I - S_h) R_h - (\lambda I - S_h) \mathcal{R}(\lambda, S_h) + (\lambda I - S_h) \mathcal{R}(\lambda, S_h) - I\| \leq \\ &\leq \limsup_{h \rightarrow 0} \|(\lambda I - S_h) R_h - (\lambda I - S_h) \mathcal{R}(\lambda, S_h)\| + \limsup_{h \rightarrow 0} \|(\lambda I - S_h) \mathcal{R}(\lambda, S_h) - I\| \leq \\ &\leq \limsup_{h \rightarrow 0} \|\lambda I - S_h\| \|R_h - \mathcal{R}(\lambda, S_h)\| \leq 0. \end{aligned}$$

Analogously we can prove that $\lim_{h \rightarrow 0} \|R_h(\lambda I - S_h) - I\| = 0$. \square

Proposition 18. Let $\{S_h\} \in C_b((0, 1], B(X))$, $\lambda \in r(\{S_h\})$ and $\{\mathcal{R}(\lambda, S_h)\} \in C_b((0, 1], B(X))$ such that

$$\lim_{h \rightarrow 0} \|(\lambda I - S_h) \mathcal{R}(\lambda, S_h) - I\| = \lim_{h \rightarrow 0} \|\mathcal{R}(\lambda, S_h)(\lambda I - S_h) - I\| = 0.$$

Then

$$\limsup_{h \rightarrow 0} \|\mathcal{R}(\lambda, S_h)\| \neq 0.$$

Proof. Suppose that $\limsup_{h \rightarrow 0} \|\mathcal{R}(\lambda, S_h)\| = 0$. Since

$$1 = \|I\| \leq \|(\lambda I - S_h) \mathcal{R}(\lambda, S_h) - I\| + \|(\lambda I - S_h) \mathcal{R}(\lambda, S_h)\|$$

And taking into account that $\{S_h\} \in C_b((0, 1], B(X))$, it follows that

$$\begin{aligned} 1 &\leq \limsup_{h \rightarrow 0} \|(\lambda I - S_h) \mathcal{R}(\lambda, S_h) - I\| + \limsup_{h \rightarrow 0} \|(\lambda I - S_h) \mathcal{R}(\lambda, S_h)\| \leq \\ &\leq \limsup_{h \rightarrow 0} \|\lambda I - S_h\| \|\mathcal{R}(\lambda, S_h)\| \leq \left(|\lambda| + \limsup_{h \rightarrow 0} \|S_h\| \right) \limsup_{h \rightarrow 0} \|\mathcal{R}(\lambda, S_h)\| = 0, \end{aligned}$$

contradiction. \square

Proposition 19. Let $\{S_h\} \in C_b((0, 1], B(X))$. If $\lambda, \mu \in r(\{S_h\})$ such that there are $\{\mathcal{R}(\lambda, S_h)\}, \{\mathcal{R}(\mu, S_h)\} \in C_b((0, 1], B(X))$ with property $\lim_{h \rightarrow 0} \|\mathcal{R}(\lambda, S_h) - \mathcal{R}(\mu, S_h)\| = 0$, then $\lambda = \mu$.

Proof. For $\lambda \in r(\{S_h\})$ let $\{\mathcal{R}(\lambda, S_h)\} \in C_b((0, 1], B(X))$ such that

$$\lim_{h \rightarrow 0} \|(\lambda I - S_h) \mathcal{R}(\lambda, S_h) - I\| = \lim_{h \rightarrow 0} \|\mathcal{R}(\lambda, S_h) (\lambda I - S_h) - I\| = 0$$

and for $\mu \in r(\{S_h\})$ let $\{\mathcal{R}(\mu, S_h)\} \in C_b((0, 1], B(X))$ such that

$$\lim_{h \rightarrow 0} \|(\mu I - S_h) \mathcal{R}(\mu, S_h) - I\| = \lim_{h \rightarrow 0} \|\mathcal{R}(\mu, S_h) (\mu I - S_h) - I\| = 0.$$

If

$$\lim_{h \rightarrow 0} \|\mathcal{R}(\lambda, S_h) - \mathcal{R}(\mu, S_h)\| = 0,$$

Having in view Proposition 17, we obtain

$$\lim_{h \rightarrow 0} \|(\mu I - S_h) \mathcal{R}(\lambda, S_h) - I\| = \lim_{h \rightarrow 0} \|\mathcal{R}(\lambda, S_h) (\mu I - S_h) - I\| = 0.$$

Hence

$$\begin{aligned} & \limsup_{h \rightarrow 0} \|(\lambda I - S_h) \mathcal{R}(\lambda, S_h) - (\mu I - S_h) \mathcal{R}(\lambda, S_h)\| \leq \\ & \leq \limsup_{h \rightarrow 0} \|(\lambda I - S_h) \mathcal{R}(\lambda, S_h) - I\| + \lim_{h \rightarrow 0} \|(\mu I - S_h) \mathcal{R}(\lambda, S_h) - I\| = 0. \end{aligned}$$

Therefore

$$|\lambda - \mu| \limsup_{h \rightarrow 0} \|\mathcal{R}(\lambda, S_h)\| = 0$$

And according to Proposition 18 ($\limsup_{h \rightarrow 0} \|R(\lambda, S_h)\| \neq 0$) it follows $\lambda = \mu$. \square

Lemma 20. *If two bounded families $\{S_h\}, \{T_h\} \in C_b((0, 1], B(X))$ are asymptotically equivalent and there is $\{R_h(\lambda)\} \in C_b((0, 1], B(X))$ such that*

$$\lim_{h \rightarrow 0} \|(\lambda I - S_h) R_h(\lambda) - I\| = \lim_{h \rightarrow 0} \|R_h(\lambda) (\lambda I - S_h) - I\| = 0,$$

then

$$\lim_{h \rightarrow 0} \|(\lambda I - T_h) R_h(\lambda) - I\| = \lim_{h \rightarrow 0} \|R_h(\lambda) (\lambda I - T_h) - I\| = 0.$$

Proof. Since the two families $\{S_h\}, \{T_h\} \in C_b((0, 1], B(X))$ are asymptotically equivalent, i.e. $\lim_{h \rightarrow 0} \|S_h - T_h\| = 0$, we have

$$\begin{aligned} & \limsup_{h \rightarrow 0} \|(\lambda I - T_h) R_h(\lambda) - I\| = \\ & = \limsup_{h \rightarrow 0} \|(\lambda I - T_h) R_h(\lambda) - (\lambda I - S_h) R_h(\lambda) + (\lambda I - S_h) R_h(\lambda) - I\| \leq \\ & \leq \limsup_{h \rightarrow 0} \|(\lambda I - T_h) R_h(\lambda) - (\lambda I - S_h) R_h(\lambda)\| + \limsup_{h \rightarrow 0} \|(\lambda I - S_h) R_h(\lambda) - I\| = \\ & = \limsup_{h \rightarrow 0} \|T_h R_h(\lambda) - S_h R_h(\lambda)\| \leq \limsup_{h \rightarrow 0} \|T_h - S_h\| \|R_h(\lambda)\| \leq 0. \end{aligned}$$

\square

Remark 21. Since $B_\infty = C_b((0, 1], B(X)) / C_0((0, 1], B(X))$ is a Banach algebra, then make sense

$$r(\{\dot{S}_h\}) = \left\{ \lambda \in \mathbb{C} \mid \exists \{\dot{R}_h\} \in B_\infty \text{ a.. } (\lambda\{\dot{I}\} - \{\dot{S}_h\})\{\dot{R}_h\} = \{\dot{I}\} = \{\dot{R}_h\}(\lambda\{\dot{I}\} - \{\dot{S}_h\}) \right\}$$

and

$$Sp(\{\dot{S}_h\}) = \mathbb{C} \setminus r(\{\dot{S}_h\}).$$

Let $\{S_h\} \in C_b((0, 1], B(X))$ and $\lambda \in r(\{S_h\})$. Fie $\{\mathcal{R}(\lambda, S_h)\} \in C_b((0, 1], B(X))$ such that

$$\lim_{h \rightarrow 0} \|(\lambda I - S_h) \mathcal{R}(\lambda, S_h) - I\| = \lim_{h \rightarrow 0} \|\mathcal{R}(\lambda, S_h) (\lambda I - S_h) - I\| = 0.$$

By Proposition 17, it results that for any $\{\mathcal{R}'(\lambda, S_h)\} \in \{\mathcal{R}(\lambda, S_h)\}$, we have

$$\lim_{h \rightarrow 0} \|(\lambda I - S_h) \mathcal{R}'(\lambda, S_h) - I\| = \lim_{h \rightarrow 0} \|\mathcal{R}'(\lambda, S_h) (\lambda I - S_h) - I\| = 0.$$

Moreover, for every $\{S'_h\} \in \{\dot{S}_h\}$, by Lemma 20 we have

$$\lim_{h \rightarrow 0} \|(\lambda I - S'_h) \mathcal{R}(\lambda, S_h) - I\| = \lim_{h \rightarrow 0} \|\mathcal{R}(\lambda, S_h) (\lambda I - S'_h) - I\| = 0.$$

Therefore every representative of class $\{\mathcal{R}(\lambda, S_h)\} \in B_\infty$ is an "inverse" for any representative of class $\{\dot{S}_h\}$.

Theorem 22. Let $\{S_h\} \in C_b((0, 1], B(X))$. Then

$$Sp(\{\dot{S}_h\}) = Sp(\{S_h\}).$$

Proof. Let $\lambda \in r(\{\dot{S}_h\})$. Then there is $\{\dot{R}_h\} \in B_\infty$ such that

$$(\lambda\{\dot{I}\} - \{\dot{S}_h\})\{\dot{R}_h\} = \{\dot{I}\} = \{\dot{R}_h\}(\lambda\{\dot{I}\} - \{\dot{S}_h\}).$$

Taking into account the algebraic relations of the Banach algebra B_∞ , it results

$$\{\dot{I}\} = (\lambda\{\dot{I}\} - \{\dot{S}_h\})\{\dot{R}_h\} = \{\lambda I - S_h\}\{\dot{R}_h\} = \{(\lambda I - S_h)R_h\}.$$

Therefore $\{(\lambda I - S_h)R_h - I\} \in B_\infty$, i.e.

$$\lim_{h \rightarrow 0} \|(\lambda I - S_h)R_h - I\| = 0.$$

Analogously we can show that $\lim_{h \rightarrow 0} \|R_h(\lambda I - S_h) - I\| = 0$. Then $\lambda \in r(\{S_h\})$.

Conversely, let $\lambda \in r(\{S_h\})$. Then there is $\{\mathcal{R}(\lambda, S_h)\} \in C_b((0, 1], B(X))$ such that

$$\lim_{h \rightarrow 0} \|(\lambda I - S_h) \mathcal{R}(\lambda, S_h) - I\| = \lim_{h \rightarrow 0} \|\mathcal{R}(\lambda, S_h) (\lambda I - S_h) - I\| = 0.$$

Let $\{R_h\} \in \{\mathcal{R}(\lambda, S_h)\}$. Then

$$\lim_{h \rightarrow 0} \|(\lambda I - S_h)R_h - I\| = \lim_{h \rightarrow 0} \|R_h(\lambda I - S_h) - I\| = 0$$

and $\{(\lambda I - S_h)R_h - I\}, \{R_h(\lambda I - S_h) - I\} \in B_\infty$, i.e.

$$(\lambda\{\dot{I}\} - \{\dot{S}_h\})\{\dot{R}_h\} = \{(\lambda I - S_h)R_h\} = \{\dot{I}\}$$

and

$$\{\dot{R}_h\} \left(\lambda \{\dot{I}\} - \{\dot{S}_h\} \right) = \{R_h(\lambda \dot{I} - S_h)\} = \{\dot{I}\}.$$

Therefore $\lambda \in r(\{\dot{S}_h\})$.

□

Remark 23. Let $\{S_h\} \in C_b((0, 1], B(X))$ and $\lambda \in r(\{S_h\})$. Then there is $\{\mathcal{R}(\lambda, S_h)\} \in C_b((0, 1], B(X))$ such that

$$\lim_{h \rightarrow 0} \|(\lambda I - S_h) \mathcal{R}(\lambda, S_h) - I\| = \lim_{h \rightarrow 0} \|\mathcal{R}(\lambda, S_h)(\lambda I - S_h) - I\| = 0.$$

if and only if

$$\left(\lambda \{\dot{I}\} - \{\dot{S}_h\} \right) \{\mathcal{R}(\lambda, S_h)\} = \{\dot{I}\} = \{\mathcal{R}(\lambda, S_h)\} \left(\lambda \{\dot{I}\} - \{\dot{S}_h\} \right).$$

Proposition 24. Let $\{S_h\}, \{T_h\} \in C_b((0, 1], B(X))$ be two families. If $\lim_{h \rightarrow 0} \|T_h S_h - S_h T_h\| = 0$, then $\lim_{h \rightarrow 0} \|R(\lambda, T_h) S_h - S_h R(\lambda, T_h)\| = 0$, for any $\lambda \in r(\{T_h\})$.

Proof. If $\lambda \in r(\{T_h\})$, then there is $\{\mathcal{R}(\lambda, T_h)\} \in C_b((0, 1], B(X))$ such that

$$\lim_{h \rightarrow 0} \|(\lambda I - T_h) \mathcal{R}(\lambda, T_h) - I\| = \lim_{h \rightarrow 0} \|\mathcal{R}(\lambda, T_h)(\lambda I - T_h) - I\| = 0.$$

Therefore

$$\lim_{h \rightarrow 0} \|T_h S_h - S_h T_h\| = 0 \Leftrightarrow \{\dot{S}_h\} \{\dot{T}_h\} = \{\dot{T}_h\} \{\dot{S}_h\} \Leftrightarrow$$

$$\{\dot{S}_h\} \{\mathcal{R}(\lambda, T_h)\} = \{\mathcal{R}(\lambda, T_h)\} \{\dot{S}_h\} \Leftrightarrow \lim_{h \rightarrow 0} \|R(\lambda, T_h) S_h - S_h R(\lambda, T_h)\| = 0.$$

□

Remark 25. i) Let $\{S_h\}, \{T_h\} \in C_b((0, 1], B(X))$ such that S_h is asymptotically equivalent with T_h , $\forall h \in (0, 1]$. Then

$$Sp(T_h) = Sp(S_h), \forall h \in (0, 1].$$

ii) Let $\{S_h\}, \{T_h\} \in C_b((0, 1], B(X))$ be asymptotically equivalent. Then

$$Sp(\{T_h\}) = Sp(\{S_h\}).$$

Theorem 26. Let $\{S_h\}, \{T_h\} \in C_b((0, 1], B(X))$ be two asymptotically quasinilpotent equivalent families. Then

$$Sp(\{T_h\}) = Sp(\{S_h\}).$$

Proof. Let $\lambda \in r(\{\dot{T}_h\})$. Then there is $\{\mathcal{R}(\dot{\lambda}, T_h)\} \in B_\infty$ such that

$$\left(\lambda\{\dot{I}\} - \{\dot{T}_h\}\right) \{\mathcal{R}(\dot{\lambda}, T_h)\} = \{\mathcal{R}(\dot{\lambda}, T_h)\} \left(\lambda\{\dot{I}\} - \{\dot{T}_h\}\right) = \{\dot{I}\}.$$

Since B_∞ is a Banach algebra, the map $\lambda \mapsto \{\mathcal{R}(\dot{\lambda}, T_h)\} : r(\{\dot{T}_h\}) \rightarrow B_\infty$ is analytic. Let $D_1 = \{\lambda \in \mathbb{C} \mid |\lambda - \lambda_0| \leq r_1\} \subset r(\{\dot{T}_h\})$ and $D_0 = \{\lambda \in \mathbb{C} \mid |\lambda - \lambda_0| \leq r_0\}$ with $r_1 > r_0$.

Set

$$\{R_n(\dot{\lambda})\} = \frac{1}{n!} \frac{d^n}{d\lambda^n} \{\mathcal{R}(\dot{\lambda}, T_h)\}, \forall n \in \mathbb{N},$$

and

$$\{R(\dot{\lambda})\} = \sum_{n \in \mathbb{N}} \frac{(-1)^n}{n!} \{(S_h - T_h)^{[n]}\} \{R_n(\dot{\lambda})\}.$$

If we set $M_1 = \sup_{\mu \in D_1} \|\{\mathcal{R}(\dot{\mu}, T_h)\}\|$, it follows that $\|\{R_n(\dot{\lambda})\}\| \leq \frac{r_1 M_1}{(r_1 - r_0)^{n+1}}$.

Deriving the relation $\left(\lambda\{\dot{I}\} - \{\dot{T}_h\}\right) \{\mathcal{R}(\dot{\lambda}, T_h)\} = \{\dot{I}\}$ by n times, we obtain

$$\left(\lambda\{\dot{I}\} - \{\dot{T}_h\}\right) \frac{d^n}{d\lambda^n} \{\mathcal{R}(\dot{\lambda}, T_h)\} = -n \frac{d^{n-1}}{d\lambda^{n-1}} \{\mathcal{R}(\dot{\lambda}, T_h)\}.$$

Moreover, since

$$\begin{aligned} \{(T_h - S_h)^{[n+1]}\} &= \{T_h(T_h - S_h)^{[n]} - (T_h - S_h)^{[n]}S_h\} = \\ &= \{\dot{T}_h\} \{(T_h - S_h)^{[n]}\} - \{(T_h - S_h)^{[n]}\} \{\dot{S}_h\}, \forall n \in \mathbb{N}, \end{aligned}$$

we have

$$\begin{aligned} \left(\lambda\{\dot{I}\} - \{\dot{S}_h\}\right) \{R(\dot{\lambda})\} &= \{\lambda I - S_h\} \sum_{n \in \mathbb{N}} \frac{(-1)^n}{n!} \{(S_h - T_h)^{[n]}\} \{R_n(\dot{\lambda})\} = \\ &= \sum_{n \in \mathbb{N}} \frac{(-1)^n}{n!} \{\lambda I - S_h\} \{(S_h - T_h)^{[n]}\} \{R_n(\dot{\lambda})\} = \\ &= \sum_{n \in \mathbb{N}} \frac{(-1)^n}{n!} \{(\lambda I - S_h)(S_h - T_h)^{[n]}\} \{R_n(\dot{\lambda})\} = \\ &= \sum_{n \in \mathbb{N}} \frac{1}{n!} \left\{ \left(((\lambda I - S_h) - (\lambda I - T_h))^{[n+1]} + ((\lambda I - S_h) - (\lambda I - T_h))^{[n]} (\lambda I - T_h) \right) \right\} \{R_n(\dot{\lambda})\} = \\ &= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n!} \{(S_h - T_h)^{[n+1]}\} \{R_n(\dot{\lambda})\} + \left(\lambda\{\dot{I}\} - \{\dot{T}_h\}\right) \{\mathcal{R}(\dot{\lambda}, T_h)\} - \\ &\quad - \sum_{n=1}^{\infty} \frac{(-1)^n}{n-1!} \{(S_h - T_h)^{[n]}\} \left(\lambda\{\dot{I}\} - \{\dot{T}_h\}\right) \frac{d^n}{d\lambda^n} \{\mathcal{R}(\dot{\lambda}, T_h)\} = \end{aligned}$$

$$\begin{aligned}
&= \sum_{n=0} \frac{(-1)^{n+1}}{n!} \left\{ (S_h - T_h)^{[n+1]} \right\} \{R_n(\lambda)\} + \left(\lambda \{I\} - \{T_h\} \right) \{ \mathcal{R}(\lambda, T_h) \} - \\
&\quad - \sum_{n=1} \frac{(-1)^n}{n-1!} \left\{ (S_h - T_h)^{[n]} \right\} \frac{d^{n-1}}{d\lambda^{n-1}} \{ \mathcal{R}(\lambda, T_h) \} = \\
&= \sum_{n=0} \frac{(-1)^{n+1}}{n!} \left\{ (S_h - T_h)^{[n+1]} \right\} \{R_n(\lambda)\} + \left(\lambda \{I\} - \{T_h\} \right) \{ \mathcal{R}(\lambda, T_h) \} - \\
&\quad - \sum_{n=1} \frac{(-1)^n}{n-1!} \left\{ (S_h - T_h)^{[n]} \right\} \{R_{n-1}(\lambda)\}.
\end{aligned}$$

Therefore $\lambda \in r(\{S_h\})$.

Analogously we can prove the other inclusion. By Theorem 22, it results that

$$Sp(\{T_h\}) = Sp(\{\dot{T}_h\}) = Sp(\{\dot{S}_h\}) = Sp(\{S_h\}).$$

□

Theorem 27. Let $\{T_h\} \in C_b((0, 1], B(X))$ and Ω be an open set which contains $\bigcup_{h \in (0, 1]} Sp(T_h)$. Then for any analytic function $f : \Omega \rightarrow \mathbb{C}$ we have

$$Sp(\{f(T_h)\}) = f(Sp(\{T_h\})).$$

Proof. If $\{T_h\} \in C_b((0, 1], B(X))$, then there is a $M < \infty$ such that $\|T_h\| \leq M$, $\forall h \in (0, 1]$. Therefore $Sp(T_h) \subset D(0, M) \forall h \in (0, 1]$, so that $\bigcup_{h \in (0, 1]} Sp(T_h)$ is a bounded set.

" \supseteq " Let $f : \Omega \rightarrow \mathbb{C}$ be an analytic function and $\lambda \in Sp(\{T_h\})$. For $\xi \in \Omega$, we define the function

$$g(\xi) = \begin{cases} \frac{f(\xi) - f(\lambda)}{\xi - \lambda}, & \xi \neq \lambda \\ f'(\lambda), & \xi = \lambda \end{cases}.$$

Hence $g : \Omega \rightarrow \mathbb{C}$ is analytic and

$$f(T_h) - f(\lambda)I = g(T_h)(T_h - \lambda I) = (T_h - \lambda I)g(T_h),$$

for any $h \in (0, 1]$.

We suppose that $f(\lambda) \in r(\{f(T_h)\})$. Then there is $\{\mathcal{R}(f(\lambda), f(T_h))\} \subset B(X)$ such that

$$\begin{aligned}
&\lim_{h \rightarrow 0} \|(f(\lambda)I - f(T_h))\mathcal{R}(f(\lambda), f(T_h)) - I\| = \\
&= \lim_{h \rightarrow 0} \|\mathcal{R}(f(\lambda), f(T_h))(f(\lambda)I - f(T_h)) - I\| = 0.
\end{aligned}$$

Having in view the last relation, we have

$$\begin{aligned}
&\lim_{h \rightarrow 0} \|(T_h - \lambda I)g(T_h)\mathcal{R}(f(\lambda), f(T_h)) - I\| = \\
&= \lim_{h \rightarrow 0} \|\mathcal{R}(f(\lambda), f(T_h))g(T_h)(T_h - \lambda I) - I\| = 0. (*)
\end{aligned}$$

Since

$$g(T_h)T_h = T_h g(T_h),$$

for any $h \in (0, 1]$, according to the properties of holomorphic functional calculi it follows

$$g(T_h)f(T_h) = f(T_h)g(T_h),$$

for every $h \in (0, 1]$. Applying Proposition 21, we obtain

$$\lim_{h \rightarrow 0} \|g(T_h)\mathcal{R}(f(\lambda), f(T_h)) - \mathcal{R}(f(\lambda), f(T_h))g(T_h)\| = 0.$$

Hence

$$\begin{aligned} & \lim_{h \rightarrow 0} \|g(T_h)\mathcal{R}(f(\lambda), f(T_h))(T_h - \lambda I) - I\| = \\ &= \lim_{h \rightarrow 0} \|g(T_h)\mathcal{R}(f(\lambda), f(T_h))(T_h - \lambda I) - \mathcal{R}(f(\lambda), f(T_h))g(T_h)(T_h - \lambda I) + \\ & \quad + \mathcal{R}(f(\lambda), f(T_h))g(T_h)(T_h - \lambda I) - I\| \leq \\ & \leq \lim_{h \rightarrow 0} \|g(T_h)\mathcal{R}(f(\lambda), f(T_h)) - \mathcal{R}(f(\lambda), f(T_h))g(T_h)\| \|T_h - \lambda I\| + \\ & \quad + \lim_{h \rightarrow 0} \|\mathcal{R}(f(\lambda), f(T_h))g(T_h)(T_h - \lambda I) - I\| = 0. (**) \end{aligned}$$

From (*) and (**), it results

$$\begin{aligned} & \lim_{h \rightarrow 0} \|(T_h - \lambda I)g(T_h)\mathcal{R}(f(\lambda), f(T_h)) - I\| = \\ &= \lim_{h \rightarrow 0} \|g(T_h)\mathcal{R}(f(\lambda), f(T_h))(T_h - \lambda I) - I\| = 0, \end{aligned}$$

so $\lambda \in r(\{T_h\})$, contradiction with $\lambda \in Sp(\{T_h\})$. Therefore $f(\lambda) \in Sp(\{f(T_h)\})$.
 "⊆" Let $\lambda \in Sp(\{f(T_h)\})$. If $\lambda \notin f(Sp(\{T_h\}))$, then $\lambda \neq f(\xi)$ for any $\xi \in Sp(\{T_h\})$.
 Let Ω' an open neighborhood $\bigcup_{h \in (0,1]} Sp(T_h)$ and

$$h(\xi) = \frac{1}{f(\xi) - \lambda},$$

for every $\xi \in \Omega'$. Then h is an analytic function and applying the holomorphic functional calculi, we obtain

$$h(T_h)(f(T_h) - \lambda I) = (f(T_h) - \lambda I)h(T_h) = I,$$

for any $h \in (0, 1]$. Therefore $\lambda \in r(f(T_h))$, for any $h \in (0, 1]$. Since $\bigcap_{h \in (0,1]} r(f(T_h)) \subseteq r(\{f(T_h)\})$ (Remark 3.2 i)), it follows $\lambda \in r(\{f(T_h)\})$, contradiction with $\lambda \in Sp(\{f(T_h)\})$. Hence $\lambda \in f(Sp(\{T_h\}))$. □

Definition 28. A family $\{U_h\} \subset L(X)$ is calling asymptotic quasinilpotent operator if

$$\lim_{n \rightarrow \infty} \limsup_{h \rightarrow 0} \|U_h^n\|^{\frac{1}{n}} = 0.$$

Theorem 29. *A family $\{U_h\} \in C_b((0, 1], B(X))$ is an asymptotic quasinilpotent operator if and only if $Sp(\{U_h\}) = \{0\}$.*

Proof. Let $\{U_h\} \in C_b((0, 1], B(X))$ be an asymptotic quasinilpotent operator. Then $\{U_h\}$ is asymptotically spectral equivalent with $\{0\}_{h \in (0, 1]} \in C_b((0, 1], B(X))$. By Theorem 26 it follows that

$$Sp(\{U_h\}) = Sp(\{0\}) = \{0\}.$$

Consequently, suppose that $Sp(\{U_h\}) = \{0\}$. By Theorem 22, we have

$$Sp(\{\dot{U}_h\}) = Sp(\{U_h\}) = \{0\}.$$

Then the spectral radius of $\{\dot{U}_h\}$, which we will call from now $r_{sp}(\{\dot{U}_h\})$, is zero. Since

$$r_{sp}(\{\dot{U}_h\}) = \lim_{n \rightarrow \infty} \left\| (\{\dot{U}_h\})^n \right\|^{\frac{1}{n}},$$

it follows that

$$\lim_{n \rightarrow \infty} \left\| (\{\dot{U}_h\})^n \right\|^{\frac{1}{n}} = 0.$$

But, on the other hand, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \left\| (\{\dot{U}_h\})^n \right\|^{\frac{1}{n}} &= \lim_{n \rightarrow \infty} \inf_{\{U_h\} \in \{\dot{U}_h\}} \|\{U_h\}^n\|^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \inf_{\{U_h\} \in \{\dot{U}_h\}} \|\{U_h^n\}\|^{\frac{1}{n}} = \\ &= \lim_{n \rightarrow \infty} \inf_{\{U_h\} \in \{\dot{U}_h\}} \sup_{h \in (0, 1]} \|U_h^n\|^{\frac{1}{n}} \geq \lim_{n \rightarrow \infty} \inf_{\{U_h\} \in \{\dot{U}_h\}} \limsup_{h \rightarrow 0} \|U_h^n\|^{\frac{1}{n}} = \\ &= \lim_{n \rightarrow \infty} \limsup_{h \rightarrow 0} \|U_h^n\|^{\frac{1}{n}}. \end{aligned}$$

By the above relations, we obtain

$$\lim_{n \rightarrow \infty} \limsup_{h \rightarrow 0} \|U_h^n\|^{\frac{1}{n}} = 0,$$

so that $\{U_h\}$ is an asymptotic quasinilpotent operator. \square

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